

# Operator product expansion of Wilson surfaces from M5-branes

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**ABSTRACT:** The operator product expansion (OPE) of the Wilson surface operators in six-dimensional  $(2, 0)$  superconformal field theory is studied from AdS/CFT correspondence in this paper. We compute the OPE coefficients of the chiral primary operators using the M5-brane description for spherical Wilson surface operators in higher dimensional representations. We use the non-chiral M5-brane action in our calculation. We also discuss their membrane limit, and compare our results with the ones obtained from membrane description.

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## 1. Introduction

Supersymmetric Wilson loops play an important role in  $AdS_5/CFT_4$  correspondence [1, 2, 3, 4]. On the field theory side, the calculation of the expectation values of half-BPS circular Wilson loops could be reduced to the corresponding calculation in a zero-dimensional matrix model [5]. The reduction to the matrix model relies on the fact that the perturbative contributions to the expectation value are believed to be only from the rainbow graphs in Feynman gauge [5] and this is confirmed by using the conformal transformation which links the straight Wilson line and the circular Wilson loop [6]. It is remarkable that the computations using the matrix model give us the results to all orders of  $g_{YM}^2 N$  and to all orders of  $1/N$ . The dependence on  $1/N$  indicates that in order to have a good dual description of these BPS Wilson loops, one has to go beyond the free string limit and consider the string interaction on the  $AdS_5$  side.

The original  $AdS_5/CFT_4$  dictionary tells us that the dual description of the Wilson loops in  $AdS_5$  should be the fundamental strings whose worldsheet boundaries are just the paths used to define the Wilson loops in  $\mathcal{N} = 4$  Super-Yang-Mills theory[7, 8]. The on-shell classical actions of the strings give the expectation values of the Wilson loops, after

correctly including the boundary terms [9]. However, the field theory result indicates that this should not be the full story and one should go beyond the free string limit. Later on, people found that a better description for the half-BPS Wilson loop in high rank representation of gauge group is using D3-brane and/or D5-brane configurations[10, 11, 12, 13]. The D3-brane configuration gives a good description for the Wilson loops in the symmetric representation, while the D5-brane gives a good description for the ones in the antisymmetric representation. The original string picture is a good description only for the Wilson loops in the fundamental representation or low dimensional representations. The D-brane description of the Wilson loops in high dimensional representations can be understood as dielectric effect[14, 15]: due to the interaction among many coincide fundamental strings in the self-dual RR background, the strings blow up to higher dimensional D-branes. The expectation values of the Wilson loops can be computed from the action of the classical D-brane solutions in the large N limit, appropriately taken into account of the boundary terms. The computations using D-branes successfully reproduce the all-genus results from matrix model calculation[10, 11]. Furthermore D3-brane description of some 1/4-BPS Wilson loops was given in [16] and the D-brane description of 1/2-BPS Wilson-'t Hooft operators was given in [17]. Some further studies of higher rank Wilson loops using matrix model can be found in [18, 19].

Another interesting issue on Wilson loops is to calculate their OPE. When we probe the Wilson loop from a distance much larger than the size of the loop, this Wilson loop operator can be expanded as a linear combination of local operators. When the local operator is a chiral primary operator, the OPE coefficient can be computed either from the correlation function of two Wilson loops or from the correlation of the Wilson loop with this operator [20]. According to AdS/CFT correspondence, in the large N and large  $g_{YM}^2 N$  limit this OPE coefficient can be computed from the coupling to the string worldsheet corresponding to the Wilson loop of the supergravity mode corresponding to the chiral primary operator [20]. When the Wilson loop operator is in high dimensional representation, the OPE coefficients can be computed from the coupling to the corresponding D-branes of the supergravity modes [21].

Motivated by the success in the Wilson loop case, we would like to consider its cousin in six-dimensional field theory in the framework of  $AdS_7/CFT_6$  correspondence. Here  $CFT_6$  is a six-dimensional superconformal field theory with  $(2,0)$ -supersymmetries. Its field content is of a tensor multiplet, including a 2-form  $B_{\mu\nu}$ , four fermions and five scalars; the field strength of this 2-form is (anti)-self-dual. The strong version of the  $AdS_7/CFT_6$  correspondence claims that this field theory is dual to the M-theory on the background  $AdS_7 \times S^4$ . This correspondence was obtained by considering  $N$  coinciding M5-branes in M-theory. The low energy limit of the worldvolume theory is the above six-dimensional  $A_{N-1}, (2,0)$  superconformal field theory [22, 23]. The near horizon limit of the supergravity solution corresponding to these M5-branes will give  $AdS_7 \times S^4$  background with 4-form flux. Similar to the  $AdS_5/CFT_4$  case, this near horizon limit led Maldacena to propose the above correspondence [1].<sup>1</sup> Unfortunately, unlike the well-studied  $AdS_5/CFT_4$  case,

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<sup>1</sup>In [24], this correspondence was used to study the nonsupersymmetric QCD in four dimensions.

the  $AdS_7/CFT_6$  correspondence is poorly investigated, although its study could be essential for us to understand M-theory. The main obstacle is our ignorance of the mysterious superconformal field theory. Due to the existence of self-dual chiral 2-form, there is no lagrangian formulation of the theory, even though the chiral theory is still a local interacting field theory[25]. The theory has been suggested to be described by DLCQ matrix theory[26, 27]. In any sense, it has not been well understood. The AdS/CFT correspondence supplies a new tool to probe this nontrivial six-dimensional field theory. The weak version of the correspondence says that the large  $N$  limit of the  $(2,0)$  field theory is dual to 11D supergravity on  $AdS_7 \times S^4$ [1]. The chiral primary operators and the corresponding supergravity modes in this case were studied in [28]. Some correlation functions of local operators were computed in [29]. These local operators were also studied using M5-brane action[30].

In this six-dimensional superconformal field theory, the natural cousin of Wilson loop operator is Wilson surface operator, a non-local operator of dimension two. This operator could be formally defined as [31]

$$W_0(\Sigma) = \exp i \int_{\Sigma} B^+. \quad (1.1)$$

Here  $\Sigma$  is a two-dimensional surface. From AdS/CFT correspondence, the Wilson surface operator should correspond to a membrane ending on the boundary of AdS space[8, 20]. Inspired by the D-brane description of Wilson loops in higher dimensional representations, M5-brane description of the half-BPS Wilson surface operators in higher dimensional representations were studied in details in [32].<sup>2</sup> The corresponding M5-brane solutions of the covariant equations of motion have been found. Both the straight Wilson surfaces and the spherical Wilson surfaces were studied in this framework. For each case, two kinds of solution were discovered. Both of them have worldvolume of topology  $AdS_3 \times S^3$ . The  $AdS_3$  part is always in  $AdS_7$ , while the  $S^3$  part can be either in  $AdS_7$  or in  $S^4$ . Analogizing the D-brane description of the Wilson loops, we expect the first case describe the Wilson surface in the symmetric representation, while the second solution describe the Wilson surface in the anti-symmetric representation. The expectation value of the Wilson surface should be given by the action of the membrane or the M5-brane. Both actions are divergent[20, 32]. For the straight Wilson surface, there is only quadratic divergence, but for the spherical Wilson surface, there are both quadratic and logarithmic divergences. The logarithmic divergence comes from conformal anomaly of the surface operator[35]. The existence of logarithmic divergence indicates that the expectation value of Wilson surface may not be well-defined. Despite of this fact, the OPE coefficients of the Wilson surface operators are still well-defined. In [29], the OPE coefficients of the chiral primary operators are computed using the membrane solution found in [20]. The strategy is similar to the Wilson loop case: one may treat the membrane as the source for the supergravity fields in the bulk. The OPE coefficients could be read off from the coupling to the membrane of the bulk supergravity modes corresponding to the chiral primary operators.

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<sup>2</sup>Similar brane configurations for straight Wilson surface are discussed in [33] in Pasti-Sorokin-Tonin (PST) formalism as well. The self-dual string soliton in  $AdS_4 \times S^7$  spacetime is discussed in [33, 34].

The main subject of this paper is to compute the OPE coefficients for the Wilson surface operator in higher dimensional representation using the M5-brane solutions mentioned above. We compute these OPE coefficients from the correlation functions of the Wilson surface with the chiral primary operators. Instead of taking the membrane as source, we take the M5-brane as the source and study its response to the bulk gravity modes. Unlike the cases of D-brane and M2-brane, the dynamics of M5-brane is much more subtler. Various actions of M5-branes are given in [36]-[41]. In this paper we will use the non-chiral action in [41] to compute the OPE coefficients. The virtue of this action is that we need not to introduce any auxiliary fields.

The paper is organized as follows. In section 2, we review the computations of the OPE coefficients using membrane solution. Section 3 is devoted to a very brief review of the non-chiral action of M5-brane. The computations using M5-brane solution is present in the following two sections. Section 4 is for the symmetric case and section 5 is for the anti-symmetric case. We end with the conclusion and discussions. We put the technical details about the variation of dual six-form gauge potential  $\delta C_6$  in the appendix.

## 2. Review of the OPE of the Wilson surface in the fundamental presentation

In this paper, we only consider the spherical Wilson surface operators. When we probe the Wilson surface from a distance quite larger than its radius  $r$ , the operator product expansion of the Wilson surface operators could be:

$$W(S) = \langle W(S) \rangle (1 + \sum_{i,n} c_i^n r^{\Delta_i^n} \mathcal{O}_i^n), \quad (2.1)$$

where  $\mathcal{O}_i^n$  are operators with conformal weights  $\Delta_i^n$ . Here we use  $\mathcal{O}_i^0$  to denote the  $i$ -th primary field and  $\mathcal{O}_i^n$  for  $n > 0$  to denote its conformal descendants. The OPE coefficients of the chiral primary operator  $\mathcal{O}_i^0$  can be obtained from the  $r/L$  expansion of the correlation function of the Wilson surface with this chiral primary operator,

$$\frac{\langle W(S) \mathcal{O}_i^0 \rangle}{\langle W(S) \rangle} = c_i^0 \frac{r^{\Delta_i^0}}{L^{2\Delta_i^0}} + \sum_{m>0} c_i^m r^{\Delta_i^m} \langle \mathcal{O}_i^m \mathcal{O}_i^0 \rangle, \quad (2.2)$$

where  $L$  is the distance from the Wilson surface to the local operator, and we have assume that the local operators have been normalized.

These operators should be bosonic and  $S_N$  symmetric, since they should have the same symmetry property of the Wilson surface. Based on the experience from the supersymmetric Wilson loops in  $\mathcal{N} = 4$  SYM, the half-BPS Wilson surface should also coupled to the five scalars. This coupling is determined by a vector  $\tilde{\theta}^I(s)$  in  $S^4$  [20]. We consider the case when  $\tilde{\theta}^I(s) = \tilde{\theta}^I$  is a constant, i. e. a fixed point in  $S^4$ . Then the R-symmetry group is broken from  $SO(5)$  to  $SO(4)$ . The local operators which appear in the OPE of the Wilson surface should also be in the representation of  $SO(5)$  whose decomposition includes singlet of  $SO(4)$ . In this paper, we will compute the OPE coefficient of the operator  $\mathcal{O}_\Delta$  in the rank  $k$  symmetric, traceless representation of  $SO(5)$ . This operator satisfy the above

constraints and is a chiral primary operator of dimension  $\Delta = 2k$  [27]. The dimension of this operator is protected by supersymmetries.

## 2.1 Review of the corresponding supergravity modes

In the following, we would like to review the supergravity modes corresponding to this chiral primary operators. To do this, we would like to first review the  $AdS_7 \times S^4$  solution of 11d supergravity. This solution is maximally supersymmetric.

The bosonic equations of motion of 11d supergravity are <sup>3</sup>:

$$R_{\underline{mn}} = \frac{1}{2 \times 3!} H_{\underline{mpqr}} H_{\underline{n}}{}^{\underline{pqr}} - \frac{1}{6 \times 4!} g_{\underline{mn}} H_{\underline{pqrs}} H^{\underline{pqrs}}, \quad (2.3)$$

$$0 = \partial_{\underline{m}} (\sqrt{-g} H^{\underline{mnpq}}) + \frac{1}{2 \times (4!)^2} \epsilon^{\underline{m}_1 \dots \underline{m}_8 \underline{n} \underline{p} \underline{q}} H_{\underline{m}_1 \dots \underline{m}_4} H_{\underline{m}_5 \dots \underline{m}_8}. \quad (2.4)$$

And the metric and background 4-form flux of  $AdS_7 \times S^4$  are

$$\begin{aligned} ds^2 &= \frac{1}{y^2} (dy^2 - dt^2 + dx^2 + dr^2 + r^2 d\Omega_3^2) + \frac{1}{4} d\Omega_4^2 \\ H_4 &= \frac{3}{8} \sin^3 \zeta_1 \sin^2 \zeta_2 \sin \zeta_3 d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4 \end{aligned} \quad (2.5)$$

where  $d\Omega_3^2$  is the metric of unit  $S^3$  and  $d\Omega_4^2$  is the metric of unit  $S^4$ . The 4-form field strength fills in  $S^4$ , and  $\zeta_i$  ( $i = 1, 2, 3, 4$ ) are the angular coordinates in  $S^4$ . We have rescaled the radius of  $AdS_7$  to be 1, then the radius of  $S^4$  is 1/2. From the  $AdS_7/CFT_6$  duality, we know that

$$l_p = (8\pi N)^{-\frac{1}{3}}, \quad (2.6)$$

where  $l_p$  is the Planck constant in eleven dimension. The 4-form field strength  $H_4$  and its Hodge dual 7-form field strength  $H_7$  are related to the corresponding gauge potentials  $C_3$  and  $C_6$  by

$$\begin{aligned} H_4 &= dC_3, \\ H_7 &= dC_6 + \frac{1}{2} C_3 \wedge H_4. \end{aligned} \quad (2.7)$$

Now we consider the fluctuation around the above background to get the states of 11d supergravity in this background [42, 43, 44]. We can decompose the fluctuated metric as

$$G_{\underline{mn}} = g_{\underline{mn}} + h_{\underline{mn}}, \quad (2.8)$$

where  $g_{\underline{mn}}$  is the background metric,  $h_{\underline{mn}}$  is the fluctuations. The fluctuation of the three form gauge potential is

$$\delta C_{\underline{mnp}} = a_{\underline{mnp}} \quad (2.9)$$

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<sup>3</sup>We use the following notation:  $m, n, \dots$  refer to the coordinate indices of  $AdS_7 \times S^4$ ,  $\mu, \nu, \dots$  refer to the coordinate indices of the  $AdS_7$  part,  $\alpha, \beta, \dots$  refer to ones of the  $S^4$  part, and the underline indices refer to target space ones.

We first decompose  $h_{\underline{\alpha}\underline{\beta}}$  into the trace part and the traceless part:

$$h_{\underline{\alpha}\underline{\beta}} = h_{(\underline{\alpha}\underline{\beta})} + \frac{1}{4}h_2 g_{\underline{\alpha}\underline{\beta}}. \quad (2.10)$$

Then we decompose  $h_{\underline{\mu}\underline{\nu}}$  as

$$h_{\underline{\mu}\underline{\nu}} = h'_{(\underline{\mu}\underline{\nu})} + \left(\frac{h'}{7} - \frac{h_2}{5}\right) g_{\underline{\mu}\underline{\nu}}. \quad (2.11)$$

Here  $(\underline{mn})$  indicates that we take the symmetric traceless part.

In the gauge defined by

$$\nabla^\alpha h_{(\underline{\alpha}\underline{\beta})} = \nabla^\alpha h_{\underline{\alpha}\underline{\mu}} = \nabla^\alpha a_{\underline{\alpha}\underline{mn}} = 0, \quad (2.12)$$

$h', h_2, h_{(\underline{\alpha}\underline{\beta})}, h'_{(\underline{\mu}\underline{\nu})}$  and  $a_{\underline{m}_1\underline{m}_2\underline{m}_3}$  have the following expansion:

$$\begin{aligned} h' &= \sum_I h'^I Y^I, \quad h_2 = \sum_I h_2^I Y^I, \\ h_{(\underline{\alpha}\underline{\beta})} &= \sum_I \phi^I Y_{(\underline{\alpha}\underline{\beta})}^I, \quad h'_{(\underline{\mu}\underline{\nu})} = \sum_I h'_{(\underline{\mu}\underline{\nu})}^I Y^I, \end{aligned} \quad (2.13)$$

and

$$a_{\underline{\alpha}\underline{\beta}\underline{\gamma}} = \sum_I 6\sqrt{2}\epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} b^I \nabla^{\underline{\delta}} Y^I. \quad (2.14)$$

Here  $Y^I$  and  $Y_{(\underline{\alpha}\underline{\beta})}^I$  are scalar and rank 2, symmetric traceless tensor harmonics on four-sphere with radius 1/2, respectively. They satisfy the following equations

$$\nabla^\alpha \nabla_\alpha Y^I = -4k(k+3)Y^I, \quad (2.15)$$

and

$$\nabla^\alpha \nabla_\alpha Y_{(\underline{\beta}\underline{\gamma})}^I = -4[k(k+3) - 2]Y_{(\underline{\beta}\underline{\gamma})}^I, \quad (2.16)$$

respectively<sup>4</sup>. The index  $I$  is the abbreviation of  $(l_4, \dots, l_1)$  which satisfy

$$l_4 \equiv k \geq l_3 \geq l_2 \geq |l_1|. \quad (2.17)$$

Using the above expansions, we can obtain the linearized equations of motion which we will not repeat here. The modes  $h_2$  and  $b$  satisfy a set of coupled equations of motion. The mass eigenvectors and eigenvalues are

$$s^I = \frac{k}{2k+3}[h_2^I + 32\sqrt{2}(k+3)b^I], \quad m_s^2 = 4k(k-3), \quad k \geq 2, \quad (2.18)$$

$$t^I = \frac{k+3}{2k+3}[h_2^I - 32\sqrt{2}kb^I], \quad m_t^2 = 4(k+3)(k+6), \quad k \geq 0. \quad (2.19)$$

$s^I$  transforms in the same representation of the R-symmetry group  $SO(5)$  as  $\mathcal{O}_\Delta$ , and it is the supergravity mode corresponding to  $\mathcal{O}_\Delta$  [28].

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<sup>4</sup>We use the same normalization of the harmonic functions as in [29].

Since we are only interested in the OPE coefficients of  $\mathcal{O}_\Delta$ , we can set the other modes to be zero. From  $t^I = 0$ , we get

$$h_2^I = 32\sqrt{2}kb^I, \quad (2.20)$$

so

$$s^I = 32\sqrt{2}kb^I = h_2^I. \quad (2.21)$$

Using the results in [29], we can express the fluctuation of the background in terms of  $s^I$  as:

$$h_{\underline{\alpha}\underline{\beta}}^I = \frac{1}{4}g_{\underline{\alpha}\underline{\beta}}s^I, \quad (2.22)$$

$$h_{\underline{\mu}\underline{\nu}}^I = \frac{3}{16k(2k+1)}\nabla_{(\underline{\mu}}\nabla_{\underline{\nu})}s^I - \frac{1}{14}g_{\underline{\mu}\underline{\nu}}s^I, \quad (2.23)$$

and

$$\delta C_{\underline{\alpha}\underline{\beta}\underline{\gamma}} = \sum_I \frac{3}{16k}\epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}s^I\nabla^{\underline{\delta}}Y^I. \quad (2.24)$$

## 2.2 Review of the computations of the OPE coefficients

In this subsection we will review the membrane solution corresponding to the Wilson surface in the fundamental representation in [20] and the computations of the OPE coefficients using this solution [29].

The membrane solution can be described more conveniently in the Euclidean version of  $AdS_7$  space and using the Poincaré coordinates. In this coordinate system, the metric of the  $AdS_7$  space is

$$ds^2 = \frac{1}{y^2}(dy^2 + \sum_{i=1}^6 dx_i^2). \quad (2.25)$$

Consider a spherical Wilson surface with radius  $r$  described by

$$x_1^2 + x_2^2 + x_3^2 = r^2, \quad (2.26)$$

in the boundary of  $AdS_7$ . The membrane solution corresponding to this Wilson surface can be parametrized as following:

$$\begin{aligned} x_1 &= \sqrt{r^2 - y^2} \cos \theta, \\ x_2 &= \sqrt{r^2 - y^2} \sin \theta \cos \psi, \\ x_3 &= \sqrt{r^2 - y^2} \sin \theta \sin \psi, \end{aligned} \quad (2.27)$$

where  $0 \leq y \leq r, 0 \leq \theta \leq \pi, 0 \leq \psi \leq 2\pi$ .

To compute the OPE coefficient from this membrane solutions, we need to use the action of the M2-brane. The bosonic part of this action is [45]

$$S_{M2} = T_2 \int (d\text{Vol} - \underline{C}_3), \quad (2.28)$$

where  $T_2$  is the tension of M2-brane:

$$T_2 = \frac{1}{(2\pi)^2 l_p^3} = \frac{2N}{\pi}, \quad (2.29)$$



and  $\underline{C}_3$  is the pullback of the bulk 3-form gauge potential to the worldvolume of the membrane<sup>5</sup>. Since the worldvolume of the membrane is completely embedded in the  $AdS_7$  part of the background, so the pullback of  $\delta C_3$  to the membrane worldvolume is zero. Then the only contribution is from the Nambu-Goto part of the action:

$$\delta S_{M2} = \frac{1}{2} T_2 \int d\text{Vol} g^{mn} h_{mn}. \quad (2.30)$$

After we compute the fluctuation of the action due to the supergravity modes, we write  $s^I$  as  $s^I(\vec{x}, y) = \int d^6 \vec{x}' G_\Delta(\vec{x}'; \vec{x}, y) s_0^I(\vec{x}')$ . Here

$$G_\Delta(\vec{x}'; \vec{x}, y) = c \left( \frac{y}{y^2 + |\vec{x} - \vec{x}'|^2} \right)^\Delta, \quad (2.31)$$

is the bulk-to-boundary propagator and  $c$  is the following constant:

$$c = \frac{8^{2+k}(2k-3)(2k-1)(2k+1)\Gamma(k+3/2)}{9\pi^{1/2}N^3\Gamma(k)}. \quad (2.32)$$

Then the correlation function we needed to compute is:

$$\frac{\langle W(\mathcal{S}, L) \mathcal{O}_\Delta(0) \rangle}{\langle W(\mathcal{S}, L) \rangle} \sim -\frac{1}{\mathcal{N}^I} \frac{\delta S_{M2}}{\delta s_0^I(\vec{x})}. \quad (2.33)$$

Here

$$\mathcal{N}^I = -2^{3k/2+3} \frac{(2k-3)(2k+1)}{3\pi^{1/4}N^{3/2}} \sqrt{\frac{(2k-1)\Gamma(k+1/2)}{\Gamma(k)}} \quad (2.34)$$

is used to set the normalization of the operator and this constant is fixed by requiring the coefficient of the 2-point function to be unit.

Since we only need to compute this function to the first order of  $r/L$ , we can use the following approximation for the bulk-to-boundary propagator[20]:

$$G_\Delta(\vec{x}'; \vec{x}, y) \simeq c \frac{y^\Delta}{L^{2\Delta}}. \quad (2.35)$$

From eq. (2.31), we find that to the first order of  $r/L$ , we have the following approximations:

$$\partial_\mu s^I \simeq \delta_\mu^y \frac{\Delta}{y} s^I, \quad \partial_\mu \partial_\nu s^I \simeq \delta_\mu^y \delta_\nu^y \frac{\Delta(\Delta-1)}{y^2} s^I, \quad (2.36)$$

By using this and

$$\Gamma_{\underline{\mu}\underline{\nu}}^y = y g_{\underline{\mu}\underline{\nu}} - \frac{2}{y} \delta_\mu^y \delta_\nu^y \quad (2.37)$$

in Poincaré coordinate, we get

$$h_{\underline{\mu}\underline{\nu}}^I \simeq -\frac{1}{8} g_{\underline{\mu}\underline{\nu}} s^I + \frac{3}{8} \delta_\mu^y \delta_\nu^y \frac{1}{y^2} s^I. \quad (2.38)$$

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<sup>5</sup>In this paper, we use the underline indices to denote the target space indices. We also use the underline to denote the pullback of bulk gauge potential or field strength to the worldvolume of M2-brane or M5-brane. We hope that this will not produce confusion.

From this, we have

$$\delta S_{M2} = -\frac{3T_2}{16} \int d\text{Vol} \frac{y^2}{r^2} \sum_I s^I Y^I(\tilde{\theta}). \quad (2.39)$$

By using eq. (2.33), (2.32), (2.34), (2.29), we get

$$\frac{\langle W(\mathcal{S}, L) \mathcal{O}_\Delta(0) \rangle}{\langle W(\mathcal{S}, L) \rangle} \sim -2^{(3k+1)/2} \pi^{1/4} \sqrt{\frac{\Gamma(k)}{N\Gamma(k-\frac{1}{2})}} \frac{r^\Delta}{L^{2\Delta}} Y^I(\tilde{\theta}). \quad (2.40)$$

So the OPE coefficients are<sup>6</sup>

$$c_{\text{fund.}, \Delta} = -2^{(3\Delta+2)/4} \pi^{1/4} \sqrt{\frac{\Gamma(\Delta/2)}{N\Gamma((\Delta-1)/2)}}. \quad (2.41)$$

### 3. The non-chiral action of M5-brane

Compared to D-branes, the action of M5-brane is more involved. Various actions of M5-branes were given in [36]-[41]. Different choice of action gives equivalent equations of motion [37, 41]. In this paper we will use the non-chiral action in [41] to compute the OPE coefficients. There is a 3-form field strength  $H_3$  on the worldvolume of the M5-brane. This field strength is related to a 2-form potential  $A_2$  by

$$H_3 = dA_2 - \underline{C}_3, \quad (3.1)$$

so  $H_3$  satisfies the following Bianchi identity:

$$dH_3 = -\underline{H}_4 \quad (3.2)$$

Here  $\underline{C}_3$  and  $\underline{H}_4$  are the pull-back of target space 3-form potential and 4-form field strength, respectively.

The non-chiral action is given by

$$S = S_{M5} - S_{WZ} = T_5 \int \left( \frac{1}{2} \star \mathcal{K} - Z_6 \right), \quad (3.3)$$

where

$$\mathcal{K} = 2\sqrt{1 + \frac{1}{12}H^2 + \frac{1}{288}(H^2)^2 - \frac{1}{96}H_{abc}H^{bcd}H_{def}H^{efa}}, \quad (3.4)$$

$$Z_6 = \underline{C}_6 - \frac{1}{2}\underline{C}_3 \wedge H_3, \quad (3.5)$$

and  $T_5$  is the tension of the M5-brane:

$$T_5 = \frac{1}{(2\pi)^5 l_p^6} = \frac{2N^2}{\pi^3}. \quad (3.6)$$

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<sup>6</sup>Notice that  $Y^I$  is not included in the OPE coefficients.

Here  $\underline{C}_6$  is the pull-back of target space 6-form potential. The equations of motion are obtained from the variation of the action with respect to the embedding  $z^{\underline{m}}$  and the gauge potential  $A_2$ . The equation of motion for 2-form potential is equivalent to the Bianchi identity. In addition, one have to impose the following non-linear self-duality condition [41]

$$*H_3 = \frac{\partial \mathcal{K}}{\partial H_3}, \quad (3.7)$$

by hand.

In the following two sections we will study the OPE of the Wilson surface operators using this non-chiral action. For doing this, we need to compute the variations of the action with respect to the above fluctuations of the background fields reviewed in subsection 2.1. Since we only need to compute the fluctuation to the linear order and the equations of motion are obtained from the variation of the action with respect to  $z^{\underline{m}}$  and  $A_2$ , we can set the variations of  $z^{\underline{m}}$  and  $A_2$  to be zero. Then from eq. (3.1) we get  $\delta H_3 = -\delta \underline{C}_3$ .

#### 4. OPE of the Wilson surface in the symmetric representation

In this section we will study the OPE of the Wilson surface operator in the symmetric representation by using the M5-brane solutions in [32]. We would like to compute the OPE coefficients of  $\mathcal{O}_\Delta$  by compute the correlation functions of the Wilson surface operator with  $\mathcal{O}_\Delta$ . According to the AdS/CFT correspondence, we need to study the coupling to this M5-brane of the corresponding supergravity modes  $s^I$ .

##### 4.1 Review of the M5-brane solution

First we would like to review the M5-brane solution corresponding to the spherical Wilson surface operator in the symmetric representation. As in [10], it is more convenient to make a Wick rotation in the  $AdS_7$  space and choose the coordinates such that the metric take the following form:

$$ds^2 = \frac{1}{y^2}(dy^2 + dr_1^2 + r_1^2(d\alpha^2 + \sin^2 \alpha d\beta^2) + dr_2^2 + r_2^2(d\gamma^2 + \sin^2 \gamma d\delta^2)), \quad (4.1)$$

The Wilson surface will be placed at  $r_1 = r$  and  $r_2 = 0$ . Let us change the coordinates  $(r_1, r_2, y)$  to  $(\rho, \eta, \theta)$  by the following transformation:

$$r_1 = \frac{r \cos \eta}{\cosh \rho - \sinh \rho \cos \theta}, \quad r_2 = \frac{r \sinh \rho \sin \theta}{\cosh \rho - \sinh \rho \cos \theta}, \quad y = \frac{r \sin \eta}{\cosh \rho - \sinh \rho \cos \theta}, \quad (4.2)$$

then we can rewrite the  $AdS_7$  metric as

$$ds^2 = \frac{1}{\sin^2 \eta}(d\eta^2 + \cos^2 \eta(d\alpha^2 + \sin^2 \alpha d\beta^2) + d\rho^2 + \sinh^2 \rho(d\theta^2 + \sin^2 \theta d\gamma^2 + \sin^2 \theta \sin^2 \gamma d\delta^2)). \quad (4.3)$$

Here, the coordinates take the range  $\rho \in [0, \infty), \theta, \alpha, \gamma \in [0, \pi), \beta, \delta \in [0, 2\pi), \eta \in [0, \pi/2)$ .

The worldvolume of the M5-brane has topology  $AdS_3 \times S^3$  and is completely embedded into the  $AdS_7$  part of the background geometry. We take  $(\eta, \alpha, \beta, \theta, \gamma, \delta)$  as the worldvolume

coordinates of M5-brane and assume that  $\rho$  be only the function of  $\eta$ . For the solution found in [32],  $\eta$  and  $\rho$  satisfy the following relation:

$$\sinh \rho = \kappa \sin \eta \quad (4.4)$$

so the induced metric of M5-brane worldvolume is

$$ds^2 = \frac{1}{\sin^2 \eta} \left( \frac{1 + \kappa^2}{1 + \kappa^2 \sin^2 \eta} d\eta^2 + \cos^2 \eta (d\alpha^2 + \sin^2 \alpha d\beta^2) \right) + \kappa^2 (d\theta^2 + \sin^2 \theta d\gamma^2 + \sin^2 \theta \sin^2 \gamma d\delta^2). \quad (4.5)$$

The field strength  $H_3$  on the worldvolume is

$$H_3 = 2a \left( \frac{i}{(1 + a^2) \sin^3 \eta} \sqrt{\frac{1 + \kappa^2}{1 + \kappa^2 \sin^2 \eta}} \cos^2 \eta \sin \alpha d\eta \wedge d\alpha \wedge d\beta \right. \\ \left. + \frac{1}{1 - a^2} \kappa^3 \sin^2 \theta \sin \gamma d\theta \wedge d\gamma \wedge d\delta \right). \quad (4.6)$$

The equations of motion require that  $\kappa$  and  $a$  should satisfy

$$\frac{\kappa}{\sqrt{1 + \kappa^2}} = -\frac{1 - a^2}{1 + a^2} \quad (4.7)$$

#### 4.2 The computations of the OPE coefficients

To compute the coupling to the M5-brane of these supergravity modes, we should compute the variations of the action with respect to the above fluctuation of the background.

First we notice that after Wick rotation, the non-chiral action for the M5-branes take the form:

$$S = S_{M5} - S_{CS} = T_5 \int \left( \frac{1}{2} * \mathcal{K} + iZ_6 \right). \quad (4.8)$$

We decompose the fluctuation of the background metric into two parts:

$$h_{\underline{\mu}\underline{\nu}} = h_{\underline{\mu}\underline{\nu}}^{(1)} + h_{\underline{\mu}\underline{\nu}}^{(2)}, \quad (4.9)$$

where

$$h_{\underline{\alpha}\underline{\beta}}^{(1)} = \frac{1}{4} g_{\underline{\alpha}\underline{\beta}} s, \quad h_{\underline{\mu}\underline{\nu}}^{(1)} = -\frac{1}{8} g_{\underline{\mu}\underline{\nu}} s. \quad (4.10)$$

$$h_{\underline{\alpha}\underline{\beta}}^{(2)} = 0, \quad h_{\underline{\mu}\underline{\nu}}^{(2)} = \frac{3}{8} \delta_{\underline{\mu}}^y \delta_{\underline{\nu}}^y \frac{1}{y^2} s. \quad (4.11)$$

Here  $s = \sum_I s^I Y^I$ .

First we compute the variation of the action with respect to the first part of the fluctuation of the metric. Let us define

$$\delta^{(i)} = h_{\underline{mn}}^{(i)} \frac{\delta}{\delta g_{\underline{mn}}}, \quad i = 1, 2. \quad (4.12)$$

From equation (4.10), we get the first part of the fluctuation of the induced metric as

$$h_{\underline{\mu}\underline{\nu}}^{(1)} = -\frac{1}{8} g_{\underline{\mu}\underline{\nu}} s, \quad (4.13)$$

Furthermore, we have

$$\delta^{(1)} g^{\mu\nu} = \frac{1}{8} g^{\mu\nu} s. \quad (4.14)$$

Since the M5-brane is completely embedded in the  $AdS_7$ , we have

$$\begin{aligned} \delta^{(1)} \sqrt{\det g_{\mu\nu}} &= \frac{1}{2} \sqrt{\det g_{\mu\nu}} g^{\mu\nu} h_{\mu\nu}^{(1)} \\ &= -\frac{3}{8} s \sqrt{\det g_{\mu\nu}}. \end{aligned} \quad (4.15)$$

From

$$H^2 = 6(H_{\eta\alpha\beta} H^{\eta\alpha\beta} + H_{\theta\gamma\delta} H^{\theta\gamma\delta}) = 6(g^{\eta\eta} g^{\alpha\alpha} g^{\beta\beta} H_{\eta\alpha\beta}^2 + g^{\theta\theta} g^{\gamma\gamma} g^{\delta\delta} H_{\theta\gamma\delta}^2), \quad (4.16)$$

we get

$$\delta^{(1)} H^2 = 6 \cdot 3 \cdot \frac{1}{8} s (H_{\eta\alpha\beta} H^{\eta\alpha\beta} + H_{\theta\gamma\delta} H^{\theta\gamma\delta}) = \frac{3}{8} s H^2. \quad (4.17)$$

Similarly, by using

$$H_{mnp} H^{npq} H_{qrs} H^{rsm} = 12((H_{\eta\alpha\beta} H^{\eta\alpha\beta})^2 + (H_{\theta\gamma\delta} H^{\theta\gamma\delta})^2) \quad (4.18)$$

we get

$$\delta^{(1)} (H_{mnp} H^{npq} H_{qrs} H^{rsm}) = \frac{3}{4} s H_{mnp} H^{npq} H_{qrs} H^{rsm} \quad (4.19)$$

Since

$$\mathcal{K} = 2\sqrt{1 + \frac{1}{12} H^2 + \frac{1}{288} (H^2)^2 - \frac{1}{96} H_{mnp} H^{npq} H_{qrs} H^{rsm}}, \quad (4.20)$$

we have

$$\delta^{(1)} \mathcal{K} = 0. \quad (4.21)$$

We note that this result is valid for any of the M5-brane solutions completely embedded in the  $AdS_7$  space. From the above results we get

$$\delta^{(1)} (\sqrt{\det g_{\mu\nu}} \mathcal{K}) = -\frac{3}{8} s \sqrt{\det g_{\mu\nu}} \mathcal{K}. \quad (4.22)$$

Then

$$\delta^{(1)} S_{M5} = -\frac{T_5}{2} \int \frac{3}{8} s \sqrt{\det g_{\mu\nu}} \mathcal{K} d\eta d\alpha d\beta d\theta d\gamma d\delta. \quad (4.23)$$

Now we turn to compute the variation of the action with respect to the second part of the fluctuation of the background metric.

In Pioncarè coordinate, we have

$$h_{\underline{\mu}\underline{\nu}}^{(2)} = \frac{3}{8} \delta_{\underline{\mu}}^y \delta_{\underline{\nu}}^y \frac{1}{y^2} s. \quad (4.24)$$

In the new coordinate system, we have,

$$h_{\underline{\mu}\underline{\nu}}^{(2)} = \frac{3}{8} \frac{\partial y}{\partial X^{\underline{\mu}}} \frac{\partial y}{\partial X^{\underline{\nu}}} \frac{1}{y^2} s, \quad (4.25)$$

which give us

$$h_{\theta\theta}^{(2)} = h_{\underline{\theta\theta}}^{(2)} = \frac{3}{8}s \left( \frac{\sinh \rho \sin \theta}{\cosh \rho - \sinh \rho \cos \theta} \right)^2, \quad (4.26)$$

and

$$h_{\eta\eta}^{(2)} = \frac{3}{8}s \left( \frac{\cos \eta}{\sin \eta} - \frac{\kappa \cos \eta \sinh \rho - \cosh \rho \cos \theta}{\cosh \rho - \sinh \rho \cos \theta} \right)^2. \quad (4.27)$$

Similar to the previous computations, we have

$$\begin{aligned} \delta^{(2)} \sqrt{\det g_{\mu\nu}} &= \frac{1}{2} \sqrt{\det g_{\mu\nu}} g^{\mu\nu} h_{\mu\nu}^{(2)} \\ &= \frac{1}{2} \sqrt{\det g_{\mu\nu}} (g^{\eta\eta} h_{\eta\eta}^{(2)} + g^{\theta\theta} h_{\theta\theta}^{(2)}), \end{aligned} \quad (4.28)$$

$$\begin{aligned} \delta^{(2)} H^2 &= 6(\delta^{(2)} g^{\eta\eta} H_{\eta\alpha\beta} H_{\eta}^{\alpha\beta} + \delta^{(2)} g^{\theta\theta} H_{\theta\gamma\delta} H_{\theta}^{\gamma\delta}) \\ &= -6(g^{\eta\eta} h_{\eta\eta}^{(2)} H_{\eta\alpha\beta} H^{\eta\alpha\beta} + g^{\theta\theta} h_{\theta\theta}^{(2)} H_{\theta\gamma\delta} H^{\theta\gamma\delta}), \end{aligned} \quad (4.29)$$

and

$$\begin{aligned} \delta^{(2)} (H_{mnp} H^{npq} H_{qrs} H^{rsm}) &= 12(2(H_{\eta\alpha\beta} H^{\eta\alpha\beta})^2 (-g^{\eta\eta} h_{\eta\eta}^{(2)}) \\ &\quad + 2(H_{\theta\gamma\delta} H^{\theta\gamma\delta})^2 (-g^{\theta\theta} h_{\theta\theta}^{(2)})). \end{aligned} \quad (4.30)$$

Taking all these into account, we get

$$\begin{aligned} \delta^{(2)} S_{M5} &= \frac{T_5}{2} \int \sqrt{\det g_{\mu\nu}} \left\{ h_{\eta\eta}^{(2)} g^{\eta\eta} \left( \frac{\mathcal{K}}{2} - \frac{2}{\mathcal{K}} H^{\eta\alpha\beta} H_{\eta\alpha\beta} \left( \frac{1}{2} + \frac{1}{24} H^2 - \frac{1}{4} H^{\eta\alpha\beta} H_{\eta\alpha\beta} \right) \right. \right. \\ &\quad \left. \left. + (\eta, \alpha, \beta \rightarrow \theta, \gamma, \delta) \right\} d\eta d\alpha d\beta d\theta d\gamma d\delta \\ &= -\frac{T_5}{2} \int d\eta d\alpha d\beta d\theta d\gamma d\delta \frac{3}{8} s \frac{\cos^2 \eta + \sinh^2 \rho \sin^2 \theta}{(\cosh \rho - \sinh \rho \cos \theta)^2} \\ &\quad \times \frac{\kappa^2 \cos^2 \eta \sin \alpha \sin^2 \theta \sin \gamma}{\sin^3 \eta \cosh \rho}, \end{aligned} \quad (4.31)$$

where  $\kappa = \frac{1-a^2}{2|a|}$  and the explicit value of  $H_3$  have been used.

Now we begin to discuss the contributions from the fluctuation of the four-form flux. Recall that  $\delta H_3 = -\delta \underline{C}_3$ , since  $\delta C_3$  only have components in  $S^4$ , so  $\delta H_3 = 0$ . Then the contributions only come from the Chern-Simions part of the action. Since  $\delta(\underline{C}_3 \wedge H_3) = 0$ , the only contribution is from  $\delta \underline{C}_6$ .

The computations of  $\delta C_6$  is put in the appendix, the result is

$$\delta C_6 = -\frac{3}{8} C_6 \sum_I s^I Y^I. \quad (4.32)$$

Therefore

$$\delta S_{CS} = -iT_5 \int \delta \underline{C}_6 = -iT_5 \int \underline{C}_6 \left( -\frac{3}{8} s \right). \quad (4.33)$$

Using this result and eq. (4.23), we get

$$\delta^{(1)} S_{M5} - \delta S_{CS} = T_5 \int \left( \frac{1}{2} \star \mathcal{K} + i \underline{C}_6 \right) \left( -\frac{3}{8} s \right) \quad (4.34)$$

The 6-form gauge potential  $C_6$  is of the form

$$\begin{aligned} C_6 = & i \frac{\cos^3 \eta \sinh^3 \rho \sin^2 \theta \sin \alpha \sin \gamma}{\sin^6 \eta} d\rho \wedge d\alpha \wedge d\beta \wedge d\theta \wedge d\gamma \wedge d\delta \\ & - i \frac{\cos^2 \eta \sinh^2 \rho \sin^3 \theta \sin \alpha \sin \gamma}{\sin^5 \eta (\cosh \rho - \sinh \rho \cos \theta)} d\eta \wedge d\alpha \wedge d\beta \wedge d\rho \wedge d\gamma \wedge d\delta \\ & + i \frac{\cos^2 \eta \sinh^3 \rho \sin^2 \theta \sin \alpha \sin \gamma (\sinh \rho - \cos \theta \cosh \rho)}{\sin^5 \eta (\cosh \rho - \sinh \rho \cos \theta)} \\ & d\eta \wedge d\alpha \wedge d\beta \wedge d\theta \wedge d\gamma \wedge d\delta. \end{aligned} \quad (4.35)$$

On the M5-brane worldvolume,

$$\begin{aligned} \underline{C}_6 = & i \frac{\kappa^3 \cos^2 \eta \sin^2 \theta \sin \alpha \sin \gamma}{\sin^3 \eta \cosh \rho (\cosh \rho - \sinh \rho \cos \theta)} (\kappa \cosh \rho - (1 + \kappa^2) \cos \theta \sin \eta) \\ & d\eta \wedge d\alpha \wedge d\beta \wedge d\theta \wedge d\gamma \wedge d\delta \end{aligned} \quad (4.36)$$

After some calculations, we get

$$\delta^{(1)} S_{M5} - \delta S_{CS} = T_5 \int \left( -\frac{3}{8} s \right) \frac{\kappa^2 \cos^2 \eta \sin^2 \theta \sin \alpha \sin \gamma}{2 \sin^3 \eta \cosh \rho} \frac{\cosh \rho + \sinh \rho \cos \theta}{\cosh \rho - \sinh \rho \cos \theta} d\eta d\alpha d\beta d\theta d\gamma d\delta \quad (4.37)$$

From this result and eq. (4.31), we get

$$\begin{aligned} \delta S = & \delta^{(1)} S_{M5} + \delta^{(2)} S_{M5} - \delta S_{CS} = T_5 \int \frac{3}{8} s \frac{\kappa^2 \cos^2 \eta \sin^2 \theta \sin \alpha \sin \gamma}{2 \sin^3 \eta \cosh \rho} \\ & \frac{\sin^2 \eta - 2 \sinh^2 \rho \sin^2 \theta - 2}{(\cosh \rho - \sinh \rho \cos \theta)^2} d\eta d\alpha d\beta d\theta d\gamma d\delta \end{aligned} \quad (4.38)$$

Having obtained the variation of the action with respect to the fluctuation of the background fields, we can compute the correlation function of the Wilson surface operator in the symmetric representation with the chiral primary operators.

Now, we write  $s^I$  as  $s^I(\vec{x}, y) = \int d^6 \vec{x}' G_\Delta(\vec{x}'; \vec{x}, y) s_0^I(\vec{x}')$ ,

$$\begin{aligned} \frac{\langle W(\mathcal{S}, L) \mathcal{O}_\Delta(0) \rangle}{\langle W(\mathcal{S}, L) \rangle} \sim & -\frac{1}{\mathcal{N}^I} \frac{\delta S}{\delta s_0^I(\vec{x})} = -\frac{T_5}{\mathcal{N}^I} \int \frac{3}{8} s \frac{r^\Delta \kappa^{1-\Delta} \sinh^\Delta \rho \cos \eta \sin^2 \theta \sin \alpha \sin \gamma}{2 L^{2\Delta} \sin^3 \eta (\cosh \rho - \sinh \rho \cos \theta)^{\Delta+2}} \\ & \times (\sin^2 \eta - 2 \sinh^2 \rho \sin^2 \theta - 2) Y^I(\tilde{\theta}) d\rho d\alpha d\beta d\theta d\gamma d\delta. \end{aligned} \quad (4.39)$$

By using

$$\int_0^\pi d\alpha \sin \alpha \int_0^{2\pi} d\beta = \int_0^\pi d\gamma \sin \gamma \int_0^{2\pi} d\delta = 4\pi \quad (4.40)$$

and

$$\sin \eta = \kappa^{-1} \sinh \rho, \quad \cos \eta = \frac{\sqrt{\kappa^2 - \sinh^2 \rho}}{\kappa}, \quad (4.41)$$

we get

$$\frac{\langle W(\mathcal{S}, L) \mathcal{O}_\Delta(0) \rangle}{\langle W(\mathcal{S}, L) \rangle} \sim -\frac{c}{\mathcal{N}^I} \cdot (4\pi)^2 \frac{3}{8} \frac{r^\Delta}{L^{2\Delta}} T_5 \int_0^{\sinh^{-1} \kappa} \frac{1}{2} \sqrt{\kappa^2 - \sinh^2 \rho} \kappa^{3-\Delta} \sinh^{\Delta-3} \rho d\rho \int_0^\pi d\theta \frac{\sin^2 \theta (-2 + \sinh^2 \rho (\kappa^{-2} - 2 \sin^2 \theta))}{(\cosh \rho - \sinh \rho \cos \theta)^{2+\Delta}} Y^I(\tilde{\theta}). \quad (4.42)$$

So the OPE coefficient is

$$c_{S,\Delta} = 2^{3k/2+4} (k + \frac{1}{2}) \pi^{-5/4} N^{1/2} \sqrt{\frac{(2k-1)\Gamma(k+1/2)}{\Gamma(k)}} \int_0^{\sinh^{-1} \kappa} \sqrt{\kappa^2 - \sinh^2 \rho} \kappa^{3-\Delta} \sinh^{\Delta-3} \rho d\rho \int_0^\pi d\theta \frac{\sin^2 \theta (-2 + \sinh^2 \rho (\kappa^{-2} - 2 \sin^2 \theta))}{(\cosh \rho - \sinh \rho \cos \theta)^{2+\Delta}}. \quad (4.43)$$

We can perform the integral over  $\theta$  and get:

$$c_{S,\Delta} = 2^{3k/2+2} (k + \frac{1}{2}) \pi^{-1/4} N^{1/2} \sqrt{\frac{(2k-1)\Gamma(k+1/2)}{\Gamma(k)}} \int_0^{\sinh^{-1} \kappa} d\rho \sqrt{\kappa^2 - \sinh^2 \rho} \kappa^{3-\Delta} \sinh^{\Delta-3} \rho [2(\kappa^{-2} \sinh^2 \rho - 2) \exp[-(2+\Delta)\rho] {}_2F_1(3/2, 2+\Delta, 3, 1-e^{-2\rho}) - 3 \sinh^2 \rho \exp[-(2+\Delta)\rho] {}_2F_1(5/2, 2+\Delta, 5, 1-e^{-2\rho})]. \quad (4.44)$$

It would be interesting to compare our results with the OPE coefficients of Wilson surface operators in the fundamental representation computed using the membranes [29]. To do this, we should take the limit of  $\kappa \rightarrow 0$  because in this limit the  $S^3$  part of the worldvolume shrink.

In this limit, we can do the integral by substitution: we define  $t$  by using

$$\rho = (\sinh^{-1} \kappa) t, \quad 0 \leq t \leq 1 \quad (4.45)$$

then as  $\kappa \rightarrow 0$ ,

$$\sinh^{-1} \kappa \sim \kappa, \quad \rho \sim \kappa t, \quad \cosh \rho \sim 1, \quad \sinh \rho \sim \kappa t, \quad d\rho \sim \kappa dt, \quad (4.46)$$

then

$$c_{S,\Delta} = -\frac{1}{\mathcal{N}^I} \cdot 3\pi^2 c T_5 \kappa^2 \int_0^1 dt t^{\Delta-3} (t^2 - 2) \sqrt{1-t^2} \int_0^\pi d\theta \sin^2 \theta = -\frac{3\pi^3}{2} \frac{c}{\mathcal{N}^I} \frac{a^\Delta}{L^{2\Delta}} T_5 \kappa^2 \int_0^1 dt \sqrt{1-t^2} (t^{\Delta-1} - 2t^{\Delta-3}). \quad (4.47)$$

Using

$$\begin{aligned} \int_0^1 dt \sqrt{1-t^2} (t^{\Delta-1} - 2t^{\Delta-3}) &= \frac{\sqrt{\pi}}{4} \left( \frac{\Gamma(\frac{\Delta}{2})}{\Gamma(\frac{\Delta+3}{2})} - 2 \frac{\Gamma(\frac{\Delta-2}{2})}{\Gamma(\frac{\Delta+1}{2})} \right) \\ &= -\frac{\sqrt{\pi}}{4} \frac{\Delta + 4}{\Delta + 1} \frac{\Gamma(\frac{\Delta-2}{2})}{\Gamma(\frac{\Delta+1}{2})}, \end{aligned} \quad (4.48)$$



we get

$$\begin{aligned}
c_{S,\Delta} &= T_5 \frac{c}{\mathcal{N}^I} \frac{3\pi^{7/2}}{8} \kappa^2 \frac{2k+4}{2k+1} \frac{\Gamma(k-1)}{\Gamma(k+\frac{1}{2})} \\
&= -2^{(3k+3)/2} N^{1/2} \pi^{1/4} \frac{k+2}{k-1} \sqrt{\frac{\Gamma(k)}{\Gamma(k-1/2)}} \kappa^2,
\end{aligned} \tag{4.49}$$

in the  $\kappa \rightarrow 0$  limit. Now we express this result in terms of  $Q_M$ , the magnetic charge of the string soliton solution[32]. For this solution, we have  $\kappa^2 = Q_M/(8\pi N)$ , So

$$c_{S,\Delta} = -Q_M 2^{(3k-3)/2} \pi^{-3/4} \frac{k+2}{k-1} \sqrt{\frac{\Gamma(k)}{N\Gamma(k-1/2)}}. \tag{4.50}$$

We can see that in this limit the OPE coefficients is proportional to  $Q_M$ . Comparing with the results eq. (2.41) obtained from membrane, we find that the  $k$ -dependence of the OPE coefficient is different although the  $N$ -dependence is the same.

## 5. OPE of the Wilson surface in the antisymmetric representation

In this section we compute the OPE of the Wilson surface in the antisymmetric representation. As mentioned in the introduction, in this case, the worldvolume of the M5-brane still has topology  $AdS_3 \times S^3$ , where the  $S^3$  part (we sometimes call it  $\tilde{S}^3$ ) is in  $S^4$  instead of  $AdS_7$ . Some part of the calculations are similar to the previous section, while some new issues will appear here.<sup>7</sup>

### 5.1 Review of the M5-brane solution

As in section 4, we first review the M5-brane solution corresponding to the spherical Wilson surface operator in antisymmetric representation.

We begin from the Euclidean  $AdS_7$  whose metric has form (4.1). We further consider the transformation

$$y = r \cos \delta, \quad r_1 = r \sin \delta. \tag{5.1}$$

The coordinates of the  $AdS_3$  part of the M5-brane worldvolume can be chosen as  $\delta, \alpha, \beta$ . Then the  $AdS_3$  part of the induced metric of the worldvolume is

$$ds_{\text{ind}, AdS_3}^2 = \frac{1}{\cos^2 \delta} (d\delta^2 + \sin^2 \delta (d\alpha^2 + \sin^2 \alpha d\beta^2)). \tag{5.2}$$

The coordinates of the  $\tilde{S}^3$  part can be chose to be  $\zeta_2, \zeta_3, \zeta_4$  and we let  $\zeta_1$  to be fixed at a constant  $\zeta^0$ . Then the induced metric of this part is

$$ds_{\text{ind}, \tilde{S}^3}^2 = \frac{1}{4} \sin^2 \zeta^0 (d\zeta_2^2 + \sin^2 \zeta_2 d\zeta_3^2 + \sin^2 \zeta_2 \sin^2 \zeta_3 d\zeta_4^2) \tag{5.3}$$

---

<sup>7</sup>In this section, we set the vector  $\tilde{\theta}^I$  mentioned in section 2 to be  $(1, 0, 0, 0)$  by a  $SO_R(5)$  rotation. Then the corresponding angular coordination  $\zeta_1$  equals to zero.

The field strength  $H_3$  on the worldvolume is

$$H_3 = 2a \left( i \frac{1}{1+a^2} \frac{\sin^2 \delta \sin \alpha}{\cos^3 \delta} d\delta \wedge d\alpha \wedge d\beta + \frac{1}{1-a^2} \frac{\sin^3 \zeta^0}{8} \sin^2 \zeta_2 \sin \zeta_3 d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4 \right). \quad (5.4)$$

The equations of motion require  $a$  and  $\zeta^0$  should satisfy

$$a = \frac{\pm 1 + \sin \zeta^0}{\cos \zeta^0}. \quad (5.5)$$

## 5.2 The computations of the OPE coefficients

After reviewing the M5-brane solution, we now compute the OPE coefficients of the Wilson surface operators using  $AdS_7/CFT_6$  correspondence. As the computation for Wilson surfaces in the symmetric representation, we should compute the variation of the M5-brane action with respect to the fluctuation of the background fields reviewed in section 2.

For the variation with respect to the first part of the fluctuation of the metric, we have:

$$\delta^{(1)} \sqrt{\det g_{mn}} = \frac{1}{2} \sqrt{\det g_{mn}} (g^{\alpha\beta} h_{\alpha\beta}^{(1)} + g^{\mu\nu} h_{\mu\nu}^{(1)}) = \frac{1}{2} \sqrt{\det g_{mn}} \left( -\frac{3}{8} \sum_I s^I Y^I + \frac{3}{4} \sum_I s^I Y^I \right). \quad (5.6)$$

From

$$H^2 = 6(H_{\delta\alpha\beta} H^{\delta\alpha\beta} + H_{234} H^{234}), \quad (5.7)$$

and

$$H_{mnp} H^{npq} H_{qrs} H^{rsm} = 12 \cdot ((H_{\delta\alpha\beta} H^{\delta\alpha\beta})^2 + (H_{234} H^{234})^2), \quad (5.8)$$

we get

$$\delta^{(1)} H^2 = 6 \left( \frac{3}{8} \sum_I s^I Y^I H_{\delta\alpha\beta} H^{\delta\alpha\beta} - \frac{3}{4} \sum_I s^I Y^I H_{234} H^{234} \right), \quad (5.9)$$

and

$$\delta^{(1)} (H_{mnp} H^{npq} H_{qrs} H^{rsm}) = 12 \cdot \left( 2 \cdot \frac{3}{8} \sum_I s^I Y^I \cdot (H_{\delta\alpha\beta} H^{\delta\alpha\beta})^2 - 2 \cdot \frac{3}{4} \sum_I s^I Y^I (H_{234} H^{234})^2 \right). \quad (5.10)$$

So

$$\begin{aligned} & \delta^{(1)} \left( \frac{1}{12} H^2 + \frac{1}{288} (H^2)^2 - \frac{1}{96} H_{mnp} H^{npq} H_{qrs} H^{rsm} \right) \\ &= \frac{3}{8} \sum_I s^I Y^I H_{\delta\alpha\beta} H^{\delta\alpha\beta} \left( \frac{1}{2} + \frac{1}{24} H^2 - \frac{1}{4} H_{\delta\alpha\beta} H^{\delta\alpha\beta} \right) \\ & \quad - \frac{3}{4} \sum_I s^I Y^I H_{234} H^{234} \left( \frac{1}{2} + \frac{1}{24} H^2 - \frac{1}{4} H_{234} H^{234} \right). \end{aligned} \quad (5.11)$$

From

$$\mathcal{K} = 2 \sqrt{1 + \frac{1}{12} H^2 + \frac{1}{288} (H^2)^2 - \frac{1}{96} H_{mnp} H^{npq} H_{qrs} H^{rsm}}, \quad (5.12)$$

we have

$$\delta^{(1)}(\mathcal{K}) = \frac{2}{\mathcal{K}}\delta^{(1)}\left(\frac{1}{12}H^2 + \frac{1}{288}(H^2)^2 - \frac{1}{96}H_{mnp}H^{npq}H_{qrs}H^{rsm}\right). \quad (5.13)$$

Using this, we get

$$\begin{aligned} \delta^{(1)}(\sqrt{\det g_{mn}}\mathcal{K}) &= \sqrt{\det g_{mn}}\left[-\frac{3}{8}\sum_I s^I Y^I \left(\frac{\mathcal{K}}{2} - \frac{2}{\mathcal{K}}H_{\delta\alpha\beta}H^{\delta\alpha\beta}\left(\frac{1}{2} + \frac{1}{24}H^2 - \frac{1}{4}H_{\delta\alpha\beta}H^{\delta\alpha\beta}\right)\right)\right. \\ &\quad \left. + \frac{3}{4}\sum_I s^I Y^I \left(\frac{\mathcal{K}}{2} - \frac{2}{\mathcal{K}}H_{234}H^{234}\left(\frac{1}{2} + \frac{1}{24}H^2 - \frac{1}{4}H_{234}H^{234}\right)\right)\right] \\ &= \sqrt{\det g_{mn}}\left(-\frac{3}{8}\sum_I s^I Y^I \left(-\frac{1+a^2}{1-a^2}\right) + \frac{3}{4}\sum_I s^I Y^I \left(-\frac{1-a^2}{1+a^2}\right)\right) \\ &= \frac{3}{8}\sum_I s^I Y^I \sqrt{\det g_{mn}} \frac{-1+6a^2-a^4}{1-a^4}. \end{aligned} \quad (5.14)$$

Now  $y = r \cos \delta$ , so

$$h_{\delta\delta}^{(2)} = \frac{3}{8}s \left(\frac{\partial y}{\partial \delta}\right)^2 \frac{1}{y^2} = \frac{3}{8}s \frac{\sin^2 \delta}{\cos^2 \delta}. \quad (5.15)$$

Similar to the computations for the Wilson surface operators in symmetric representation, we have

$$\begin{aligned} \delta^{(2)}(\sqrt{\det g_{mn}}\mathcal{K}) &= \sqrt{\det g_{mn}}g^{\delta\delta}h_{\delta\delta}^{(2)}\left(\frac{\mathcal{K}}{2} - \frac{2}{\mathcal{K}}H_{\delta\alpha\beta}H^{\delta\alpha\beta}\left(\frac{1}{2} + \frac{1}{24}H^2 - \frac{1}{4}H_{\delta\alpha\beta}H^{\delta\alpha\beta}\right)\right) \\ &= \sqrt{\det g_{mn}}\frac{3}{8}s \sin^2 \delta \frac{a^2+1}{a^2-1} \end{aligned} \quad (5.16)$$

So from eqs. (5.14) and (5.16), we get the contribution from the fluctuation of the metric:

$$\delta_g(\sqrt{\det g_{mn}}\mathcal{K}) = \sqrt{\det g_{mn}}\frac{3}{8}\sum_I s^I Y^I \left(\frac{-1+6a^2-a^4}{1-a^4} + \sin^2 \delta \frac{a^2+1}{a^2-1}\right). \quad (5.17)$$

Now we turn to the contribution from the background flux. Unlike the symmetric case, the pullback of  $\delta C_3$  on the worldvolume is nonzero, then we will get a contribution from  $\delta H$ . In fact, from

$$\delta C_{\underline{\alpha}\underline{\beta}\underline{\gamma}} = \sum_I \frac{3}{16k}\epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} s^I \nabla^{\underline{\delta}} Y^I, \quad (5.18)$$

we get

$$\delta C_{\underline{234}} = -\frac{3}{16k}\sin^3 \zeta_1 \sin^2 \zeta_2 \sin \zeta_3 \sum_I s^I \partial_{\zeta_1} Y^I, \quad (5.19)$$

then

$$\delta \underline{C}_{234} = -\frac{3}{16k}\sin^3 \zeta^0 \sin^2 \zeta_2 \sin \zeta_3 \sum_I s^I \partial_{\zeta^0} Y^I. \quad (5.20)$$

From

$$\delta H_3 = -\delta \underline{C}_3, \quad (5.21)$$

we get

$$\delta H_{234} = \frac{3}{16k} \sin^3 \zeta^0 \sin^2 \zeta_2 \sin \zeta_3 \sum_I s^I \partial_{\zeta^0} Y^I. \quad (5.22)$$

Recall that

$$H_{234} = \frac{a}{4(1-a^2)} \sin^3 \zeta^0 \sin^2 \zeta_2 \sin \zeta_3, \quad (5.23)$$

we have<sup>8</sup>

$$\delta H_{234} = \frac{3(1-a^2)}{4ak} H_{234} \sum_I s^I \partial_{\zeta^0} Y^I. \quad (5.24)$$

From this we can easily get

$$\delta_H H^2 = 6g^{22} g^{33} g^{44} H_{234} \cdot 2\delta H_{234} = 12H_{234} H^{234} \frac{\delta H_{234}}{H_{234}}, \quad (5.25)$$

and

$$\delta_H (H_{mnp} H^{npq} H_{qrs} H^{rsm}) = 12 \cdot 4 (H_{234} H^{234})^2 \frac{\delta H_{234}}{H_{234}}. \quad (5.26)$$

Then

$$\begin{aligned} & \delta_H \left( \frac{1}{12} H^2 + \frac{1}{288} (H^2)^2 - \frac{1}{96} H_{mnp} H^{npq} H_{qrs} H^{rsm} \right) \\ &= \frac{3a(1+a^4)}{k(1-a^2)(1+a^2)^2} \sum_I s^I \partial_{\zeta^0} Y^I. \end{aligned} \quad (5.27)$$

Putting all these together, we have,

$$\delta_H (\sqrt{\det g_{mn}} \mathcal{K}) = \sqrt{\det g_{mn}} \delta_H \left( \frac{\mathcal{K}^2}{4} \right) \frac{2}{\mathcal{K}} = -\sqrt{\det g_{mn}} \frac{3a}{k(a^2+1)} \sum_I s^I \partial_{\zeta^0} Y^I. \quad (5.28)$$

Finally let us compute the variation of the Chern-Simions term. Recall that

$$Z_6 = \underline{C}_6 - \frac{1}{2} \underline{C}_3 \wedge H_3. \quad (5.29)$$

In this case,  $\delta \underline{C}_6 = 0$ , so

$$\begin{aligned} \delta Z_6 &= -\frac{1}{2} \delta \underline{C}_3 \wedge H_3 \\ &= -\frac{3ia}{16k} \frac{\sin^3 \zeta^0 \sin^2 \zeta_2 \sin \zeta_3 \sin \alpha \sin^2 \delta}{\cos^3 \delta (1+a^2)} \sum_I s^I \partial_{\zeta^0} Y^I \\ &\quad d\delta \wedge d\alpha \wedge d\beta \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4. \end{aligned} \quad (5.30)$$

The total action of M5-brane is

$$S = S_{M5} - S_{CS} = T_5 \int (\star \frac{1}{2} \mathcal{K} + iZ_6), \quad (5.31)$$

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<sup>8</sup>Here  $H_{234}$  denotes  $H_{\zeta_2 \zeta_3 \zeta_4}$ .

so we get

$$\begin{aligned}
\delta S &= T_5 \int ((\frac{1}{2}\delta_g(\sqrt{\det g_{mn}}\mathcal{K}) + \frac{1}{2}\sqrt{\det g_{mn}}\delta_H(\mathcal{K}))d\delta \wedge d\alpha \wedge d\beta \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4 \\
&\quad - \delta Z_6) \\
&= \frac{3T_5}{8} \int \frac{\sin^3 \zeta^0 \sin^2 \zeta_2 \sin \zeta_3 \sin \alpha \sin^2 \delta}{16 \cos^3 \delta} (\frac{-1 + 6a^2 - a^4}{1 - a^4} + \sin^2 \delta \frac{a^2 + 1}{a^2 - 1}) \\
&\quad \times \sum_I s^I Y^I d\delta d\alpha d\beta d\zeta_2 d\zeta_3 d\zeta_4. \\
&= \frac{3T_5}{8} \int \frac{\sin^3 \zeta^0 \sin^2 \zeta_2 \sin \zeta_3 \sin \alpha \sin^2 \delta}{16 \cos^3 \delta} (\frac{-1 + 6a^2 - a^4}{1 - a^4} + \sin^2 \delta \frac{a^2 + 1}{a^2 - 1}) \\
&\quad \times \sum_k s^k Y^k d\delta d\alpha d\beta d\zeta_2 d\zeta_3 d\zeta_4.
\end{aligned} \tag{5.32}$$

Here we have performed the integration over the 3-sphere<sup>9</sup>:

$$\int \sin^2 \zeta_2 \sin \zeta_3 \sum_I s^I Y^I d\zeta_2 d\zeta_3 d\zeta_4 = \sum_k s^k Y^{k,0}(\zeta^0). \tag{5.33}$$

As before, using

$$G_\Delta(\vec{x}'; \vec{x}, y) \simeq c \frac{y^\Delta}{L^{2\Delta}}, \quad y = r \cos \delta, \tag{5.34}$$

and

$$\int_0^\pi \sin \alpha d\alpha \int_0^{2\pi} d\beta = 4\pi, \tag{5.35}$$

we get

$$\begin{aligned}
\frac{\langle W(\mathcal{S}, L) \mathcal{O}_\Delta(0) \rangle}{\langle W(\mathcal{S}, L) \rangle} &\sim -\frac{3\pi T_5}{32} \frac{c}{\mathcal{N}^I} \sum_\Delta \frac{r^\Delta}{L^{2\Delta}} \int_0^{\pi/2} \sin^3 \zeta^0 \sin^2 \delta \cos^{\Delta-3} \delta \\
&\quad (\frac{a^4 - 6a^2 + 1}{a^4 - 1} + \sin^2 \delta \frac{a^2 + 1}{a^2 - 1}) Y^{k,0}(\zeta^0).
\end{aligned} \tag{5.36}$$

Now we perform the integration over  $\delta$ :

$$\int_0^{\pi/2} \sin^2 \delta \cos^{\Delta-3} \delta d\delta = \frac{\sqrt{\pi}}{4} \frac{\Gamma(\frac{\Delta}{2} - 1)}{\Gamma(\frac{\Delta+1}{2})}, \tag{5.37}$$

$$\int_0^{\pi/2} \sin^4 \delta \cos^{\Delta-3} \delta d\delta = \frac{\sqrt{\pi}}{4} \frac{3}{\Delta + 1} \frac{\Gamma(\frac{\Delta}{2} - 1)}{\Gamma(\frac{\Delta+1}{2})}, \tag{5.38}$$

and have

$$\begin{aligned}
\frac{\langle W(\mathcal{S}, L) \mathcal{O}_\Delta(0) \rangle}{\langle W(\mathcal{S}, L) \rangle} &\sim \mp \frac{3\pi^{3/2} T_5}{128} \frac{c}{\mathcal{N}^I} \sum_\Delta \frac{r^\Delta}{L^{2\Delta}} \sin^3 \zeta^0 (-\frac{\cos 2\zeta^0}{\sin \zeta^0} \\
&\quad + \frac{1}{\sin \zeta^0} \frac{3}{2k+1}) Y^{k,0}(\zeta^0) \frac{\Gamma(k-1)}{\Gamma(k+1/2)},
\end{aligned} \tag{5.39}$$

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<sup>9</sup>Here  $Y^{k,0}$  is the abbreviation of  $Y^{(k,0,0,0)}$ . For discussions on spherical harmonics, see, for example, [46].

after putting the explicit value of  $a$ .

The harmonic function can be written as

$$Y^{k,0}(\zeta^0) = \mathcal{N}_k C_k^{(3/2)}(x), \quad (5.40)$$

where  $x = \cos \zeta^0$ ,  $C_k^{(3/2)}(x)$  are Gegenbauer polynomials and

$$\mathcal{N}_k = \left[ \frac{\pi^{1/2} k! (2k+3)}{2^{3k+7} (k+1)(k+2) \Gamma(k+5/2)} \right]^{1/2}, \quad (5.41)$$

is obtained from the normalization of  $Y^{k,0}$ .

Therefore

$$\begin{aligned} \frac{\langle W(\mathcal{S}, L) \mathcal{O}_\Delta(0) \rangle}{\langle W(\mathcal{S}, L) \rangle} &\sim \mp \frac{3\pi^{3/2} T_5}{128} \frac{c}{\mathcal{N}^I} \sum_{\Delta} \frac{r^\Delta}{L^{2\Delta}} Y^{k,0}(0) \sin^2 \zeta^0 ((-\cos 2\zeta^0 \\ &+ \frac{3}{2k+1}) \frac{\mathcal{N}_k}{Y^{k,0}(0)} C_k^{(3/2)}(\cos \zeta^0)) \frac{\Gamma(k-1)}{\Gamma(k+1/2)}. \end{aligned} \quad (5.42)$$

From

$$Y^{k,0}(0) = \mathcal{N}_k C_k^{(3/2)}(1), \quad (5.43)$$

we have

$$\begin{aligned} \frac{\langle W(\mathcal{S}, L) \mathcal{O}_\Delta(0) \rangle}{\langle W(\mathcal{S}, L) \rangle} &\sim \mp \frac{3\pi^{3/2} T_5}{128} \frac{c}{\mathcal{N}^I} \sum_{\Delta} \frac{r^\Delta}{L^{2\Delta}} Y^{k,0}(0) \frac{1}{C_k^{(3/2)}(1)} \sin^2 \zeta^0 \\ &\times (-\cos 2\zeta^0 + \frac{3}{2k+1}) C_k^{(3/2)}(\cos \zeta^0) \frac{\Gamma(k-1)}{\Gamma(k+1/2)}. \end{aligned} \quad (5.44)$$

So the OPE coefficients is

$$\begin{aligned} c_{A,\Delta} &\sim \pm \frac{2^{(3k-5)/2} N^{1/2}}{\pi^{7/4}} \frac{C_k^{(3/2)}(\cos \zeta^0)}{C_k^{(3/2)}(1)} \sin^2 \zeta^0 \\ &\times (-\cos 2\zeta^0 + \frac{3}{2k+1}) \frac{k+1/2}{k-1} \sqrt{\frac{\Gamma(k)}{\Gamma(k-1/2)}}. \end{aligned} \quad (5.45)$$

To compare with the membrane results, we take the  $\zeta^0 \rightarrow 0$  limit in which the  $\tilde{S}^3$  will shrink. In this limit  $x \rightarrow 1$  and the OPE coefficient is equal to

$$\mp \frac{2^{(3k-5)/2} N^{1/2}}{\pi^{7/4}} (\zeta^0)^2 \sqrt{\frac{\Gamma(k)}{\Gamma(k-1/2)}} \quad (5.46)$$

The magnetic charge of string soliton solution is

$$Q_M = \frac{1}{\text{Vol}(\tilde{S}^3)} \int_{\tilde{S}^3} H = -\frac{\sin^2 \zeta^0 \cos \zeta^0}{8l_p^3}, \quad (5.47)$$

in the small  $\zeta^0$  limit, we have

$$Q_M = -\frac{(\zeta^0)^2}{8l_p^3} = -\pi N (\zeta^0)^2. \quad (5.48)$$

Then the OPE coefficients can be written as

$$\pm \frac{2^{(3k-5)/2}}{\pi^{11/4}} Q_M \sqrt{\frac{\Gamma(k)}{N\Gamma(k-1/2)}}. \quad (5.49)$$

We can see that in this limit the OPE coefficients is proportional to  $Q_M$  and the  $k$ -dependence and  $N$ -dependence of the coefficients in this limit is the same as the one in eq. (2.41) computed using M2-brane.

## 6. Conclusion and discussions

In this paper we studied the OPE of spherical half-BPS Wilson surface operators using their M5-brane description. We computed the OPE coefficients by studying the coupling to the M5-branes of the supergravity modes. In this process, we first make clear that the variation of the embedding and the 2-form gauge potential can be set to zero. Then we calculated the response of the non-chiral action of M5-brane to the bulk supergravity fields. Moreover, we had to investigate carefully the response of the Chern-Simons term in the M5-brane action to the bulk gauge potential. In the symmetric case, the three form field strength has no fluctuation and only the fluctuation of the dual 6-form gauge potential gives the contribution. On the contrary, in the antisymmetric case,  $\delta H_3$  is non-zero while  $\delta \underline{C}_6 = 0$ .

We also consider the membrane limit of our results. In this limit the  $S^3$  part of the M5-brane worldvolume shrink. We find that the OPE coefficients is proportional to  $Q_M$  which characterizes the rank of the representation. This is reminiscent of the results for the expectation values of these Wilson surfaces in [32]. There it is found that the expectation values is proportional to  $Q_M$  even before we take the membrane limit. We compare our result in this membrane limit with the results obtained from the membrane method [29]. We find that the  $N$  dependence are the same. We also find that for the Wilson surface in symmetric representation, the dependence on the dimension of the local operator is different, while in the antisymmetric case, the dependence is the same. This may be related to the nontrivial dynamics of the branes in M-theory. We hope we can come back to this point in the future.

Another subtle issue is the choice of the M5-brane action. Among the different proposals for the M5-brane action, we chose the non-chiral action since there are no auxiliary fields in this action. Although different choice of action gives equivalent equations of motion [37, 41], this does not guarantee that these actions give the same quantum dynamics. It will be interesting to compute the OPE coefficients using other actions of the M5-brane, such as the PST action and compare the results obtained from different action. In [32], the issue on choosing action also appear and make the discussions for the boundary terms quite subtle.

At this stage, quite little is known in the field theory side. Some field theory studies could be found in [47]: the conformal anomaly of abelian Wilson surface operator was calculated in  $A_1$  field theory. It is very hard to consider the nonabelian Wilson surface in field theory. It would be interesting to study the OPE of Wilson surface operators from

field theory calculation and compare the results with the ones obtained from M2-brane or M5-brane.

If we compactify the six-dimension (0, 2) SCFT on  $T^2$  with supersymmetric boundary conditions, we will obtain  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory. If the Wilson surface winds various 1-cycles of the  $T^2$ , it will give Wilson loop, 't Hooft loop or Wilson-'t Hooft loop [48, 49]. It is interesting to see if one can study this relation in the framework  $AdS/CFT$  correspondence. The relation between Wilson surface in six-dimensional SCFT and the surface operator in four-dimensional SYM is also a quite interesting subject [50, 51, 52, 53].

Another interesting subject about the Wilson surfaces in higher dimensional representation is to compute the correlation function of two Wilson surfaces, one in the fundamental representation and the other one in higher dimensional representation[54].

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## 7. Appendix: The variation of $C_6$ due to the SUGRA modes

In this appendix we compute  $\delta C_6$  due to the supergravity modes  $s^I$ . Notice that for the purpose of this paper, we can use the approximation eq. (2.36) freely here.

From

$$\delta C_{\underline{\alpha}\underline{\beta}\underline{\gamma}} = \sum_I \frac{3}{16k} \epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} s^I \nabla^{\underline{\delta}} Y^I, \quad (7.1)$$

we get

$$\delta H_{\underline{\mu}\underline{\alpha}\underline{\beta}\underline{\gamma}} = \frac{3}{16k} \sum_I \epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} \nabla^{\underline{\delta}} Y^I \nabla_{\underline{\mu}} s^I, \quad (7.2)$$

and

$$\delta H_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} = -\frac{3}{16k} \sum_I \epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} \nabla_{\underline{\epsilon}} \nabla^{\underline{\epsilon}} Y^I s^I. \quad (7.3)$$

We notice that

$$\nabla_{\underline{\epsilon}} \nabla^{\underline{\epsilon}} Y^I = -4k(k+3)Y^I, \quad (7.4)$$

so we get

$$\delta H_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} = \frac{3(k+3)}{4} \sum_I \epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} Y^I s^I, \quad (7.5)$$



considering

$$H_{\underline{\alpha}\beta\gamma\underline{\delta}} = 6\epsilon_{\underline{\alpha}\beta\gamma\underline{\delta}}, \quad (7.6)$$

$$\delta H_{\underline{\alpha}\beta\gamma\underline{\delta}} = \frac{(k+3)}{8} \sum_I H_{\underline{\alpha}\beta\gamma\underline{\delta}} Y^I s^I. \quad (7.7)$$

Since  $H_7$  is the Hodge dual of  $H_4$ :

$$(H_7)_{\underline{m}_1 \dots \underline{m}_7} = \frac{\sqrt{g}}{4!} \epsilon^{\underline{n}_1 \dots \underline{n}_4}{}_{\underline{m}_1 \dots \underline{m}_7} H_{\underline{n}_1 \dots \underline{n}_4}, \quad (7.8)$$

we get

$$\begin{aligned} (\delta H_7)_{\underline{\mu}_1 \dots \underline{\mu}_6 \underline{\alpha}} &= \frac{\sqrt{g}}{4!} 4 \epsilon^{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\alpha}_3 \underline{\mu}}{}_{\underline{\mu}_1 \dots \underline{\mu}_6 \underline{\alpha}} \delta H_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\alpha}_3 \underline{\mu}} \\ &= \frac{3}{16k} \epsilon_{\underline{\mu}_1 \dots \underline{\mu}_6} \sum_I \nabla_{\underline{\alpha}} Y^I \nabla^{\underline{\mu}} s^I \\ &\simeq \frac{3}{8} y \epsilon_{y \underline{\mu}_1 \dots \underline{\mu}_6} \sum_I s^I \nabla_{\underline{\alpha}} Y^I, \end{aligned} \quad (7.9)$$

and

$$\begin{aligned} (\delta H_7)_{\underline{\mu}_1 \dots \underline{\mu}_7} &= \frac{\sqrt{g}}{4!} \epsilon^{\underline{\alpha}_1 \dots \underline{\alpha}_4}{}_{\underline{\mu}_1 \dots \underline{\mu}_7} \delta H_{\underline{\alpha}_1 \dots \underline{\alpha}_4} + \frac{\delta \sqrt{g}}{4!} \epsilon^{\underline{\alpha}_1 \dots \underline{\alpha}_4}{}_{\underline{\mu}_1 \dots \underline{\mu}_7} H_{\underline{\alpha}_1 \dots \underline{\alpha}_4} \\ &\quad + \left( \frac{\sqrt{g}}{4!} \delta g^{\underline{\alpha}_1 \underline{\beta}_1} g^{\underline{\alpha}_2 \underline{\beta}_2} g^{\underline{\alpha}_3 \underline{\beta}_3} g^{\underline{\alpha}_4 \underline{\beta}_4} \epsilon_{\underline{\beta}_1 \dots \underline{\beta}_4 \underline{\mu}_1 \dots \underline{\mu}_7} \delta H_{\underline{\alpha}_1 \dots \underline{\alpha}_4} \right. \\ &\quad \left. + \text{three other terms from } \delta g^{\underline{\alpha}_i \underline{\beta}_i}, i = 2, 3, 4 \right). \end{aligned} \quad (7.10)$$

By using

$$\begin{aligned} \delta \sqrt{g} &= \frac{1}{2} \sqrt{g} g^{mn} \delta g_{mn} = \frac{1}{2} \sqrt{g} (g^{\underline{\alpha}\beta} \delta g_{\underline{\alpha}\beta} + g^{\underline{\mu}\nu} \delta g_{\underline{\mu}\nu}) \\ &= \frac{1}{4} \sqrt{g} s, \end{aligned} \quad (7.11)$$

and

$$\delta g^{\underline{\alpha}\beta} = -\frac{1}{4} g^{\underline{\alpha}\beta} s, \quad (7.12)$$

we get

$$(\delta H_7)_{\underline{\mu}_1 \dots \underline{\mu}_7} = \frac{k-3}{8} (H_7)_{\underline{\mu}_1 \dots \underline{\mu}_7} \sum_I s^I Y^I \quad (7.13)$$

From eq. (2.7) and  $\delta(C_3 \wedge H_4) = 0$ , we get  $d\delta C_6 = \delta H_7$ . We can choose

$$\delta C_6 = -\frac{3}{8} C_6 \sum_I s^I Y^I, \quad (7.14)$$

which lead to the above  $\delta H_7$ .

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